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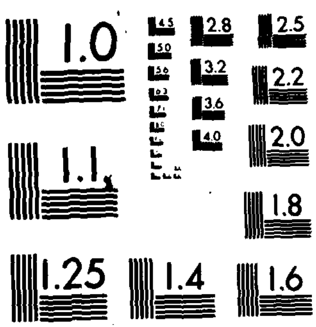

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HYDRODYNAMIC INTERACTIONS BETWEEN PARTICLES IN LOW-REYNOLDS-NUMBER FLOW: A MODULAR APPROACH

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ABSTRACT

A modular method for calculating hydrodynamic interactions between particles in low-Reynolds-number flow has been constructed by using multipole expansion solutions for the reflection field. The approach is made possible by the use of Faxen laws in relating the multipole moment to the incident field. The method is illustrated and checked by recalculating known expressions for the resistance and mobility tensors for two spheres. The method can be readily generalized to handle three-particle (or n-particle) interactions as shown in a following paper. New forms of the Faxen laws for prolate spheroids are given and will form the basis for other papers on spheroid-spheroid and spheroid-wall hydrodynamic interactions. The important result is that "first-reflection" solutions can be readily calculated even in cases where the ambient velocity field is obtained by a numerical procedure. These results, as asymptotic (far-field) solutions, furnish a check for more robust codes. In addition, these ~~they~~ are important on their own since these ~~they~~ provide crucial information for the renormalization theory used in suspension rheology.

AMS (MOS) Subject Classifications: 76D05, 35Q10

Key Words: Faxen law, hydrodynamic interaction, low Reynolds number, spheroids.

Work Unit Number 2 - Physical Mathematics

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SIGNIFICANCE AND EXPLANATION

The calculation of hydrodynamic interactions between particles is needed for the understanding and control of many natural and manufacturing processes, for instance, those involving sedimentation, colloidal stability or suspension rheology. In these applications the external forces, torques and dipole moments on the particles are known a priori and the problem is to calculate the resulting translational and rotational motions. In practice, since the governing differential equation requires knowledge of these motions for the boundary condition, one has to solve first for the forces, torques and dipole moment in a collection of translational and rotational problems and then invert to obtain the desired motions.

The purpose of this paper is to show that these problems can be solved directly. Explicit calculations and comparisons with other techniques are shown for interactions between spheres. The first step towards the corresponding calculation for spheroids (which arise in the modeling of suspension of nonspherical particles, e.g. clay/water systems) are also given.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

HYDRODYNAMIC INTERACTIONS BETWEEN PARTICLES
IN LOW-REYNOLDS-NUMBER FLOW:
A MODULAR APPROACH

Sangtae Kim

1. INTRODUCTION

Hydrodynamic interactions (HI) appear in the analysis of many different problems such as kinetic theory of polymers, mechanics of suspensions, and the convection of particles in porous media. In this series of papers, a modular approach to hydrodynamic interactions between particles of arbitrary shapes in the creeping flow regime (vanishing Reynolds number) is introduced and applied to problems in each of these areas. The approach will be illustrated by extending the literature on HI in two directions: interactions involving axisymmetric particles (prolate spheroids) and interactions involving more than two particles (three spheres in a uniform stream).

The governing equations for the fluid (viscosity μ) velocity, $\underset{\sim}{v}$ and pressure, p are taken as

$$-\nabla p + \mu \nabla^2 \underset{\sim}{v} = 0 \quad , \tag{1.1}$$

with boundary conditions at the surface of the particle

$$\underset{\sim}{v} = \underset{\sim}{U} + \underset{\sim}{\omega} \times \underset{\sim}{x}$$

where $\underset{\sim}{U}$ and $\underset{\sim}{\omega}$ are the particle translational and rotational velocities. The geometry is either that of a number of particles in a fluid of infinite extent as shown in Figure 1, a single particle in a domain with boundaries as shown in Figure 2, or a combination of the two.

The approach is modular in the sense that the HI solution is obtained by a two step procedure -- a calculation of properties intrinsic to each interacting particle, followed by a method for integrating these modules. The power of the method is due to the fact that once the first step has been accomplished, the second step follows as a "back of the envelope" calculation. As illustrated in later sections, this feature leads to considerable savings in human computation, particularly for the "mobility problems" for the calculation of the motion of particles under imposed forces and torques.

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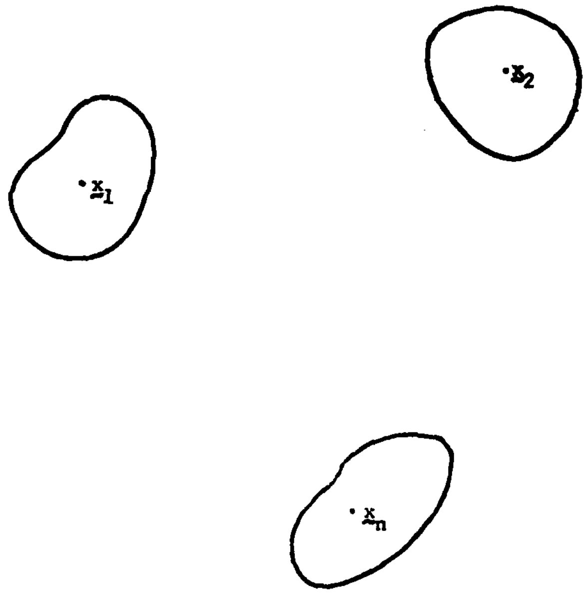

Figure 1 - Particles in an infinite domain.

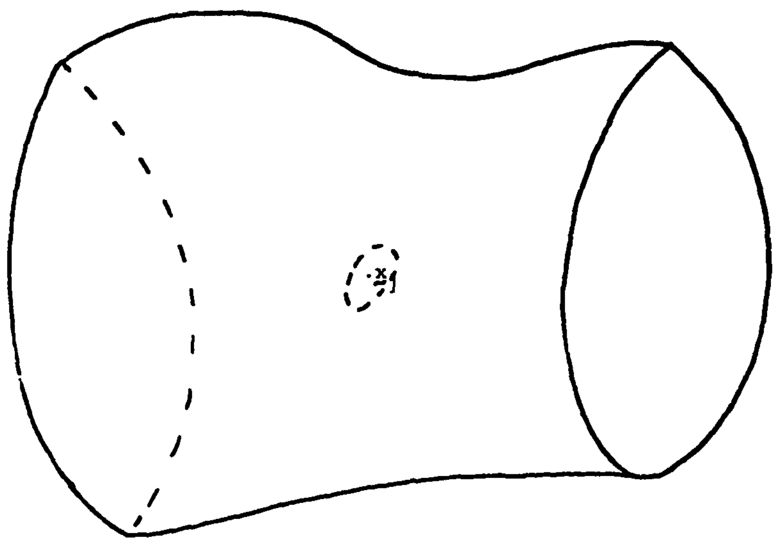

Figure 2 - Particles in a bounded domain.

The method is a generalization of the HI calculation for spheres using the method of reflections as given in Happel and Brenner (1965), and thus finds similar utility and limitations. The limitations are imposed by the complexity of the algebraic manipulations for the higher order contributions. Therefore, accurate solutions to problems involving strong interactions as defined by Ganatos et al (1980) are precluded. However, the method gives accurate results for weak interactions and thus one immediate application is its utility in the testing of more robust numerical codes in this limiting case. In addition, there are fundamental problems in suspension mechanics where the leading order terms in HI as determined from the method of reflections plays an important role (for examples on suspensions of spheres, see Batchelor 1972, Batchelor and Green 1972b, and Hinch 1977). The analogous calculations for particles of arbitrary shape can be accomplished using the techniques presented here.

The work is presented in four parts, starting with this introductory paper. In part II, the method is used to calculate the interaction between two prolate spheroids, which in turn leads to a generalization of the Rotne-Prager-Yamakawa potential used in the kinetic theory of polymers. In part III, calculations for the hydrodynamic interaction between three spheres illustrate the utility of the method for multi-particle systems. The interaction between a convected particle and the solid matrix of a porous medium is the subject of part IV.

The following section is an exposition on the method of reflections, as used in this series of papers. The multipole expansion solution for the velocity field around the particle is introduced. A procedure for calculating these multipole moments, for particles of arbitrary shape, is shown in Section 3. It is a variation of the procedure for deriving the "Faxen laws" discussed in earlier works (Faxen 1922, 1927, Brenner 1964b, Batchelor and Green 1972a, Rallison 1978). The present procedure exploits certain reciprocity relationships between the Faxen laws and velocity representations in certain associated flow fields. The method is illustrated by deriving new and possibly more useful forms of the Faxen laws for prolate spheroids.

In Section 4 and 5, the "module integration" idea is applied to various problems involving HI between spheres. Most of the results presented in these sections are already in the literature. However, the calculations, in addition to providing a test of the method, explain the surprising occurrences of zero coefficients in the expansion for the mobility tensor. Furthermore, the mobility results are obtained directly, without the usual steps of taking linear combinations of subsidiary problems.

2. THE METHOD OF REFLECTIONS

The basic solution strategy is similar to the method of reflections as given in Happel and Brenner (1965). Their * and ** notation is replaced with a system of subscripts 1 and 2 because the latter are more readily generalized for multi-sphere interaction problems. The original aspect of this work is the development and use of Faxen laws which greatly facilitate the method.

For two widely separated particles, with centers of reference at $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, the zero-th order solution for the velocity field is simply the sum of the disturbance solutions for each particle in isolation, i.e. without hydrodynamic interaction. In the present notation, these single-particle solutions are denoted $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$. Since the no-slip boundary conditions on each particle are violated by the velocity field emanating from the other particle, we correct the solution by adding on two new fields that cancel these discrepancies. However, any field that helps with the boundary condition at one particle will upset matters at the other particle, hence we get a sequence of velocity fields comprising an iterative approximation.

At each reflection, the fields that are created to help satisfy the boundary conditions are known as reflection fields. The field that violated the boundary condition (hence forcing the creation of the reflection field) is called the incident field. A schematic representation of the reflection process is given in Figure 3. The following convention will be used to label the subscripts. For a reflection at particle β, (β = 1, 2) the reflected field will be labeled by adding the subscript β to the subscripts of the incident field. Note that the isolated, single-particle solutions may be

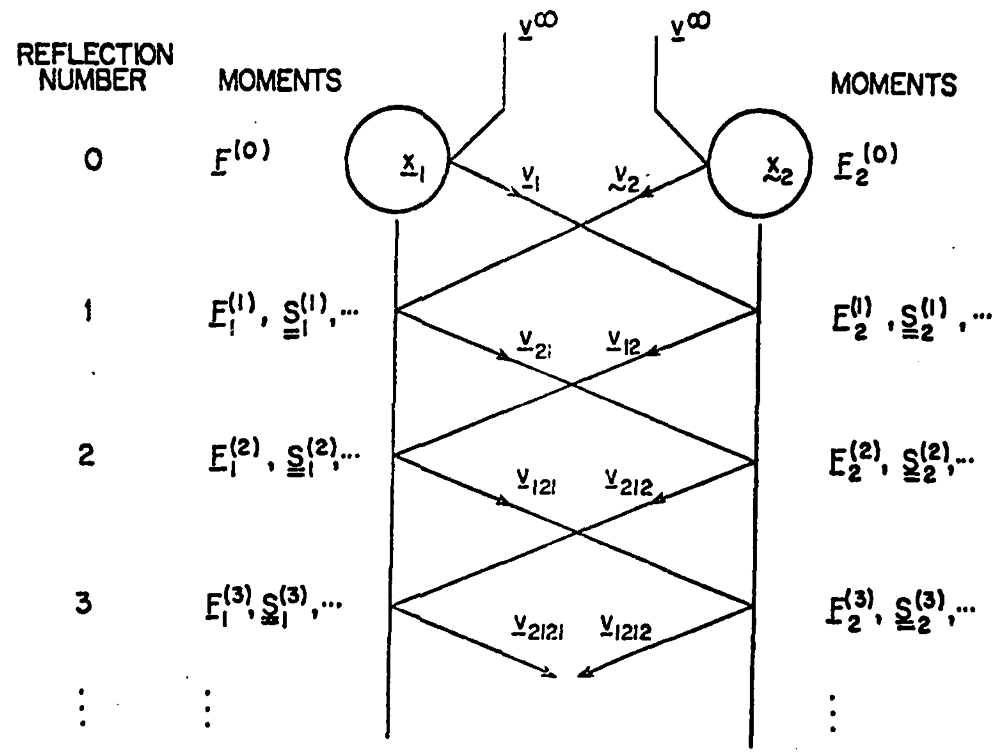

FIGURE 3.

Schematic representation of the reflection process for two spheres.

considered as reflections from the ambient velocity field $\underset{\sim}{v}^{\infty}$. This will be called the zero-th reflection in accord with the literature. The fields reflected from the single-particle solutions are the first reflections, and those reflected from the n-th reflection fields are referred to as the (n+1)-th reflection fields.

Given an incident field, we need a method for calculating the reflected field. For spheres, one can use Lamb's general solution, with Hobson's (1955) addition theorems for transforming the spherical harmonics from a coordinate centered at one sphere to one centered at the other, as shown by Happel and Brenner (1965). Instead, we present an alternative approach which can be applied to more general particle shapes. This approach is based on the Faxen laws, the integral representation for the Stokes solutions, and the multipole expansion.

An incident field $\underset{\sim}{v}^{I}(\underset{\sim}{x})$ which disturbs the boundary condition at a sphere is countered with a reflection field having the following integral representation (see Howells 1974):

$$(2.1) \qquad -\frac{1}{8\pi\mu} \oint_S (\underset{=}{\sigma}\cdot\underset{\sim}{n}) \cdot \underset{=}{I}(\underset{\sim}{x}-\underset{\sim}{x}')\, dS(\underset{\sim}{x}') \quad .$$

Here $\underset{=}{\sigma}$ is the stress evaluated at the surface, $\underset{\sim}{x}'$ is a vector from the center of reference to a point on the surface, and $\underset{=}{I}$ is the Green's dyadic or Oseen tensor given by

$$(2.2) \qquad I_{ij}(\underset{\sim}{x}) = \frac{1}{|x|}\,\delta_{ij} + \frac{1}{|x|^3}\, x_i x_j \quad .$$

The multipole expansion solution is obtained by expanding the Oseen tensor about the center of reference. Once the multipole moments, $\oint (\underset{=}{\sigma}\cdot\underset{\sim}{n})\underset{\sim}{x}'\cdots\underset{\sim}{x}'\, dS(\underset{\sim}{x}')$, are related to $\underset{\sim}{v}^{I}$, the solution can be carried out by calculating each reflection to the desired accuracy.

Such relations, known as Faxen laws are available for spheres (Faxen 1922, 1927, Batchelor and Green 1972) and for ellipsoids (Brenner 1964b, Rallison 1978). Therefore, the reflection procedure just described, can be applied readily to particles of these shapes. For nonspherical particles, however, resolution problems can arise in the module-integration step. To improve the resolution for nonspherical particles, alternate forms for the Faxen laws will be derived in the following section.

3. CONSTRUCTION OF THE FAXEN LAWS

From the previous section, it follows that the reflection fields can be constructed if the multipole moments can be expressed in terms of the incident field. This section presents the Faxen laws for the monopole and dipole moments, i.e. the force, torque and stresslet, on a particle of arbitrary shape moving with a translational velocity $\underset{\sim}{U}$ and an angular velocity $\underset{\sim}{\omega}$. Since the idea for this method came from the striking similarity between the Faxen laws and Stokes solutions for a sphere, the method will be introduced by applying it to spheres.

The Faxen laws for the force (Faxen 1922), torque (Faxen 1927) and stresslet (Batchelor and Green 1972a) on a sphere in an ambient field $\underset{\sim}{v}^{\infty}$ are compared below with the velocity fields for a stationary sphere in a uniform stream $\underset{\sim}{U}^{\infty}$, vorticity field $\underset{\approx}{\Omega}$ and rate of strain field $\underset{\approx}{E}$ (Hinch 1977):

Force:

$$\underset{\sim}{F} = 6\pi\mu a\left(1 + \frac{a^2}{6}\nabla^2\right)\underset{\sim}{v}^{\infty}(\underset{\sim}{x}_1) - 6\pi\mu a\underset{\sim}{U} \quad . \tag{3.1a}$$

Solution for uniform stream:

$$\underset{\sim}{v}(\underset{\sim}{x}) = \underset{\sim}{U}^{\infty} - 6\pi\mu a\underset{\sim}{U}^{\infty}\cdot\left(1 + \frac{a^2}{6}\nabla^2\right)\frac{\underset{\approx}{I}(\underset{\sim}{x}-\underset{\sim}{x}_1)}{8\pi\mu} \quad . \tag{3.1b}$$

Torque:

$$\underset{\sim}{T} = 4\pi\mu a^3(\nabla\times\underset{\sim}{v}^{\infty}(\underset{\sim}{x}_1) - 2\underset{\sim}{\omega}) \quad . \tag{3.2a}$$

Solution for vorticity field:

$$v(\underset{\sim}{x}) = \underset{\approx}{\Omega}^{t}\cdot\underset{\sim}{x} + 4\pi\mu a^3\underset{\approx}{\Omega}^{t}\cdot\nabla\,\frac{\underset{\approx}{I}(\underset{\sim}{x}-\underset{\sim}{x}_1)}{8\pi\mu} \quad . \tag{3.2b}$$

Stresslet:

$$\underset{\approx}{S} = \frac{20}{3}\pi\mu a^3\left(1 + \frac{a^2}{10}\nabla^2\right)\frac{1}{2}\left(\nabla\underset{\sim}{v}^{\infty} + \nabla\underset{\sim}{v}^{\infty t}\right)\Big|_{\underset{\sim}{x}=\underset{\sim}{x}_1} \quad . \tag{3.3a}$$

Solution for rate of strain field:

$$\underset{\sim}{v}(\underset{\sim}{x}) = \underset{\approx}{E}\cdot\underset{\sim}{x} + \frac{20}{3}\pi\mu a^3(\underset{\approx}{E}\cdot\nabla)\cdot\left(1 + \frac{a^2}{10}\nabla^2\right)\frac{\underset{\approx}{I}(\underset{\sim}{x}-\underset{\sim}{x}_1)}{8\pi\mu} \quad . \tag{3.3b}$$

This similarity, as noted by Hinch (1977) is rooted in the Lorentz reciprocal theorem for Stokes flow. In fact, the reciprocal theorem can be used to generate the Faxen laws as shown by Brenner (1964b) and Rallison (1978). However, this formal procedure can be streamlined by using the fundamental solutions for the velocity as shown below.

The single-sphere solution (3.1b) satisfies the boundary condition at $r = a$:

(3.4) $$6\pi\mu a(1 + \frac{a^2}{6}\nabla^2)\frac{\underline{\underline{J}}(\underset{\sim}{x}-\underset{\sim}{x})_1}{8\pi\mu}\Big|_{|\underset{\sim}{x}-\underset{\sim}{x}_1|=a} = \underline{\underline{\delta}} \quad .$$

If we go back to the integral representation for a sphere in an ambient velocity field $\underset{\sim}{v}^{\infty}(\underset{\sim}{x})$

(3.5) $$\underset{\sim}{v}(\underset{\sim}{x}) = \underset{\sim}{v}^{\infty}(\underset{\sim}{x}) - \frac{1}{8\pi\mu}\oint_S (\underline{\underline{\sigma}}\cdot\underset{\sim}{n})\cdot\underline{\underline{J}}(\underset{\sim}{x}-\underset{\sim}{x}')\, dS(\underset{\sim}{x}')$$

and then operate on both sides with the LHS of (3.4), the result is

(3.6) $$6\pi\mu a(1 + \frac{a^2}{6}\nabla^2)\underset{\sim}{v}(\underset{\sim}{x}_1) = 6\pi\mu a(1 + \frac{a^2}{6}\nabla^2)\underset{\sim}{v}^{\infty}(\underset{\sim}{x}_1) - \oint_S \underline{\underline{\sigma}}\cdot\underset{\sim}{n}\, dS(\underset{\sim}{x}') \quad .$$

In the sphere, $\underset{\sim}{v} = \underset{\sim}{U} + \underset{\sim}{\omega}\times(\underset{\sim}{x}-\underset{\sim}{x}_1)$ so that

$$6\pi\mu a(1 + \frac{a^2}{6}\nabla^2)\underset{\sim}{v}(\underset{\sim}{x}_1) = 6\pi\mu a\underset{\sim}{U} \quad .$$

The result is (3.1a), the Faxen law for the force on the sphere. Looking back, we see that the procedure is equivalent to Brenner's (1964b) approach since the integral representation is a special case of the reciprocal theorem.

The tensor identities which follow from the boundary conditions for (3.2b) and (3.3b) similarly give the Faxen laws for the torque and stresslet when applied to the integral representation.

The above protocol between single-particle solution and Faxen law is dictated by the literature -- single-particle solutions are more readily available, the Faxen laws are not. However, for new shapes, the efficient procedure is to construct the appropriate tensor identity, which then leads to both the single-particle solution and Faxen law.

For prolate spheroids, the singularity solutions of Chwang and Wu (1974, 1975) furnish the identities necessary for the derivation of the Faxen laws. In order to simplify the algebraic manipulations, their various cases have been combined by using dyadic notation, into three solutions for a stationary spheroid in a uniform stream, vorticity field and rate-of-strain field. The conversion to dyadic notation is straightforward for the uniform stream and vorticity solutions, since the solutions correspond to cases where the spheroid axis $\underset{\sim}{d}$ is either parallel or perpendicular to the relevant vector (the uniform stream or the pseudo-vector for the vorticity field). For the rate of strain field, the three cases correspond to axisymmetric straining and hyperbolic straining in planes containing and per-

pendicular to the spheroid axis. The tensorial form can be obtained either from geometrical arguments, by inspection (Batchelor and Green 1972a) or after rigorous algebraic steps (Brenner 1974).

The solution of Chwang and Wu for a stationary prolate spheroid (major semiaxis a, focal length $2c$, eccentricity $e = c/a$, and axis $\underset{\sim}{d}$) are shown below. The constants are given in Table 1.

Table 1. Constants for the velocity representation for the spheroid

i) Constants derived from Chwang and Wu (1974, 1975)

$$\alpha_1 = e^2\left\{-2e + (1+e^2)\log\left(\frac{1+e}{1-e}\right)\right\}^{-1}$$

$$\alpha_2 = 2e^2\left\{2e + (3e^2-1)\log\left(\frac{1+e}{1-e}\right)\right\}^{-1}$$

$$\gamma = (1-e^2)\left\{2e-(1-e^2)\log\left(\frac{1+e}{1-e}\right)\right\}^{-1}$$

$$\gamma_3 = (1-e^2)\left\{-2e+(1+e^2)\log\left(\frac{1+e}{1-e}\right)\right\}^{-1}$$

$$\gamma_3' = \gamma_3(1-e^2)^{-1}$$

$$\alpha_3 = 2e^2\gamma_3\left\{-2e + \log\left(\frac{1+e}{1-e}\right)\right\}\left\{2e(2e^2-3) + 3(1-e^2)\log\left(\frac{1+e}{1-e}\right)\right\}^{-1}$$

$$\alpha_3' = e^2\gamma_3'\left\{-2e+(1-e^2)\log\left(\frac{1+e}{1-e}\right)\right\}\left\{2e(2e^2-3) + 3(1-e^2)\log\left(\frac{1+e}{1-e}\right)\right\}^{-1}$$

$$\alpha_4 = 2e^2(1-e^2)\left\{2e(3-5e^2) - 3(1-e^2)^2\log\left(\frac{1+e}{1-e}\right)\right\}^{-1}$$

$$\alpha_5 = e^2\left\{6e-(3-e^2)\log\left(\frac{1+e}{1-e}\right)\right\}^{-1}$$

ii) Constants used in equations (3.7a-c), (3.8a-c)

$$a' \equiv \alpha_3 + \alpha_3' = e^2\left\{-2e+(1+e^2)\log\left(\frac{1+e}{1-e}\right)\right\}^{-1}$$

$$\gamma' \equiv \gamma_3 + \gamma_3' = (2-e^2)\left\{-2e+(1+e^2)\log\left(\frac{1+e}{1-e}\right)\right\}^{-1}$$

$$\alpha^* \equiv \alpha_3 - \alpha_3'$$

$$\gamma^* \equiv \gamma_3 - \gamma_3' = -e^2\{-2e+(1+e^2)\ \log(\frac{1+e}{1-e})\}^{-1}$$

(3.7a)

$$\underset{\sim}{v}(\underset{\sim}{x}) = \underset{\sim}{U}^\infty - \underset{\sim}{U}^\infty\cdot(\alpha_1\underset{\sim}{dd} + \alpha_2(\underset{=}{\delta} - \underset{\sim}{dd}))\ \cdot$$

$$\int_{-c}^{c} \left(1 + (c^2 - \xi^2)\ \frac{(1-e^2)}{4e^2}\ \nabla^2\right)\ \underset{=}{I}(\underset{\sim}{x}-\underset{\sim}{\xi})d\xi$$

(3.7b)

$$v_i(\underset{\sim}{x}) = \varepsilon_{ijk}\Omega_j x_k - \frac{1}{2}\ \Omega_\ell\varepsilon_{jkm}\{\gamma d_\ell d_m + \gamma'(\delta_{\ell m}-d_\ell d_m)\}\ \int_{-c}^{c}\ (c^2-\xi^2)I_{ij,k}(\underset{\sim}{x}-\underset{\sim}{\xi})d\xi$$

$$+ \Omega_\ell d_m\varepsilon_{k\ell m}d_j\alpha'\ \int_{-c}^{c}\ (c^2-\xi^2)\{1 + (c^2-\xi^2)\ \frac{(1-e^2)}{8e^2}\ \nabla^2\}\ \frac{1}{2}\ \{I_{ij,k}(\underset{\sim}{x}-\underset{\sim}{\xi}) + I_{ik,j}(\underset{\sim}{x}-\underset{\sim}{\xi})\}d\xi$$

with $\Omega_i = \frac{1}{2}\ \varepsilon_{ijk}\Omega_{jk}$

$$v_i(\underset{\sim}{x}) = E_{ij}x_j - E_{\ell m}\{\alpha_5\ \frac{1}{2}\ (d_jd_k - \underline{\tfrac{1}{3}\ \delta_{jk}})(d_\ell d_m - \underline{\tfrac{1}{3}\ \delta_{\ell m}})$$

$$= \frac{1}{4}\ \alpha^*(d_j\delta_{km}d_\ell + d_j\delta_{k\ell}d_m + \delta_{jm}d_kd_\ell + \delta_{j\ell}d_kd_m - 4d_jd_kd_\ell d_m)$$

$$+ \frac{1}{2}\ \alpha_4(\delta_{j\ell}\delta_{km} + \delta_{jm}\delta_{k\ell} - \underline{\delta_{jk}\delta_{\ell m}} + \underline{d_jd_k\delta_{\ell m}} + \underline{\delta_{jk}d_\ell d_m}$$

$$-d_j\delta_{km}d_\ell - \delta_{jm}d_kd_\ell - d_j\delta_{k\ell}d_m - \delta_{j\ell}d_kd_m + d_jd_kd_\ell d_m\}\ \times$$

$$\int_{-c}^{c}\ (c^2-\xi^2)\{1 + (c^2-\xi^2)\ \frac{(1-e^2)}{8e^2}\ \nabla^2\}I_{ij,k}(\underset{\sim}{x}-\underset{\sim}{\xi})d\xi$$

(3.7c)

$$+E_{\ell m}d_j\delta_{km}d_\ell\gamma^*\ \int_{-c}^{c}\ (c^2-\xi^2)\ \frac{1}{2}\ \{I_{ij,k}(\underset{\sim}{x}-\underset{\sim}{\xi}) - I_{ik,j}(\underset{\sim}{x}-\underset{\sim}{\xi})\}d\xi\ \ .$$

Terms underlined in (3.7c) make no contributions, but are retained for symmetry reasons. Note that the distribution of stresslets in (3.7b) and rotlets in (3.7c) are consistent with the Lorentz reciprocal theorem. Thus the Faxen laws for the force, torque and stresslet on a spheroid moving with a translational velocity $\underset{\sim}{U}$ and rotational velocity $\underset{\sim}{\omega}$ are:

$$\underset{\sim}{F} = 8\pi\mu\{\alpha_1\underset{\sim}{dd} + \alpha_2(\underset{=}{\delta} - \underset{\sim}{dd})\}\cdot\int_{-c}^{c}\{1 + (c^2-\xi^2)\ \frac{(1-e^2)}{4e^2}\ \nabla^2\}\ \underset{\sim}{v}^\infty(\underset{\sim}{\xi})d\xi$$

$$- 16\pi\mu ae\{\alpha_1\underset{\sim}{dd} + \alpha_2(\underset{=}{\delta} - \underset{\sim}{dd})\}\cdot\underset{\sim}{U}$$

(3.8a)

$$\underset{\sim}{T} = 4\pi\mu\{\gamma\underset{\sim}{dd} + \gamma'(\underset{=}{\delta} - \underset{\sim}{dd})\}\cdot\int_{-c}^{c}(c^2-\xi^2)\ \nabla\times\underset{\sim}{v}^\infty(\underset{\sim}{\xi})d\xi$$

$$+ 8\pi\mu\alpha' \underset{\sim}{d} \times \int_{-c}^{c} (c^2-\xi^2)\{1 + (c^2-\xi^2)\frac{(1-e^2)}{8e^2}\nabla^2\}\underset{\sim}{d}\cdot\underset{\approx}{e}^{\infty}(\underset{\sim}{\xi})d\xi$$

$$- \frac{32}{3}\pi\mu a^3 e^3\{\gamma\underset{\sim\sim}{dd} + \gamma'(\underset{\approx}{\delta} - \underset{\sim\sim}{dd})\}\cdot\underset{\sim}{\omega}$$

(3.8b)

$$S_{ij} = 8\pi\mu\{-\frac{1}{2}\alpha_5(d_i d_j - \frac{1}{3}\delta_{ij})(d_k d_\ell - \frac{1}{3}\delta_{k\ell})$$

$$- \frac{1}{4}\alpha^*(d_i\delta_{jk}d_\ell + d_i\delta_{j\ell}d_k + \delta_{i\ell}d_j d_k + \delta_{ik}d_j d_\ell - 4d_i d_j d_k d_\ell)$$

$$- \frac{1}{2}\alpha_4(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk} - \delta_{ij}\delta_{k\ell} + d_i d_j\delta_{k\ell} + \delta_{ij}d_k d_\ell + d_i d_j d_k d_\ell$$

$$- d_i\delta_{j\ell}d_k - d_i\delta_{jk}d_\ell - \delta_{ik}d_j d_\ell - \delta_{i\ell}d_j d_k)\}\int_{-c}^{c}(c^2-\xi^2)\{1+(c^2-\xi^2)\frac{(1-e^2)}{8e^2}\nabla^2\}e_{k\ell}^{\infty}(\underset{\sim}{\xi})d\xi$$

(3.8c) $$- 2\pi\mu\gamma^*(d_i\varepsilon_{jk\ell}d_\ell + d_j\varepsilon_{ik\ell}d_\ell)\int_{-c}^{c}(c^2-\xi^2)\{\nabla\times v^{\infty}(\underset{\sim}{\xi})\}_k d\xi \quad .$$

These expressions for the Faxen law involve integrals of the ambient velocity field and its lower order derivatives, whereas the ones developed by Brenner (1964b) and Rallison (1978) use infinite series involving derivatives of the ambient velocity field. If the ambient velocity field is derived from a numerical method, the latter approach may run into problems caused by the loss of numerical resolution (finite difference solutions) or differentiability (finite element solutions). Our integral forms yield the known differential forms if one expands $\underset{\sim}{v}$, $\nabla\times\underset{\sim}{v}$ and $\underset{\approx}{e}$ in Taylor series about the spheroid center.

For the important problem of determing the motion of a force-free and torque-free spheroid, (3.8a-c) shows that the translational and rotational velocities are:

(3.9a) $$\underset{\sim}{U} = \frac{1}{2c}\int_{-c}^{c}\{1 + (c^2-\xi^2)\frac{(1-e^2)}{4e^2}\nabla^2\}\underset{\sim}{v}^{\infty}(\underset{\sim}{\xi})d\xi$$

$$\underset{\sim}{\omega} = \frac{3}{8c^3}\int_{-c}^{c}(c^2-\xi^2)\,\nabla\times\underset{\sim}{v}^{\infty}(\underset{\sim}{\xi})d\xi$$

(3.9b)

$$+ \frac{3}{4c^3}\frac{e^2}{(2-e^2)}\int_{-c}^{c}(c^2-\xi^2)\{1 + (c^2-\xi^2)\frac{(1-e^2)}{8e^2}\nabla^2\}\underset{\sim}{d}\times\underset{\approx}{e}^{\infty}(\underset{\sim}{\xi})\cdot\underset{\sim}{d}\,d\xi \quad .$$

From (3.9b), the change in the orientation of the spheroid follows as:

$$\dot{\underset{\sim}{d}} = \frac{3}{4c^3} \int_{-c}^{c} (c^2-\xi^2) \underset{\sim}{\Omega}^{\infty}(\underset{\sim}{\xi}) \times \underset{\sim}{d} \, d\xi$$

(3.9c)

$$+ \frac{3}{4c^3} \frac{e^2}{(2-e^2)} \int_{-c}^{c} (c^2-\xi^2)\{1 + (c^2-\xi^2) \frac{(1-e^2)}{8e^2} \nabla^2\}(\underset{\approx}{e}^{\infty}(\xi)\cdot\underset{\sim}{d} - \underset{\approx}{e}^{\infty}(\underset{\sim}{\xi}):\underset{\sim}{ddd})d\xi \quad .$$

For homogeneous ambient velocity fields (constant velocity gradient) the vorticity and rate-of-strain tensors can be taken out of the integrals and (3.9b) and (3.9c) reduce to the Jeffery equations as given by Leal and Hinch (1972). The application of equations (3.8a-c) and (3.9a,b) in the method of reflections (the module integration step) will be the subject of parts II and IV.

There are two complications in the generalization of the procedure to particles of arbitrary shape. Because of the high degree of symmetry possessed by spheres and spheroids, relatively few solutions (or tensor identities) were required. The spheroid involved two solutions for the uniform stream, two for the vorticity field and three for the rate-of-strain respectively. For the general case, three streaming solutions, three vorticity solutions and five rate-of-strain solutions are required. Secondly, it is unlikely that analytical expressions will be available for these eleven single-particle solutions (or identities).

The solution technique of Youngren and Acrivos (1975) is recommended since it will lead to Faxen laws involving integrals of the ambient velocity field over the surface of the particle. They solved the integral equation for the velocity field by collocating the surface of the particle into small subdivisions. The approximants for the velocity in the uniform stream, vorticity field and rate of strain field cases,

(3.10a)

$$\underset{\sim}{v}(\underset{\sim}{x}) = \underset{\sim}{U}^{\infty} - \frac{1}{8\pi\mu} \oint_S \underset{\sim}{f}(\underset{\sim}{x}')\cdot\underset{\approx}{I}(\underset{\sim}{x}-\underset{\sim}{x}')dS(\underset{\sim}{x}')$$

$$\text{with } \underset{\sim}{f}(\underset{\sim}{x}') = \underset{\sim}{U}^{\infty}\cdot\underset{\approx}{A}(\underset{\sim}{x}') \quad ,$$

(3.10b)

$$\underset{\sim}{v}(\underset{\sim}{x}) = \underset{\sim}{\Omega}^{\infty} \times \underset{\sim}{x} - \frac{1}{8\pi\mu} \oint_S \underset{\sim}{f}(\underset{\sim}{x}')\cdot\underset{\approx}{I}(\underset{\sim}{x}-\underset{\sim}{x}')dS(\underset{\sim}{x}')$$

$$\text{with } \underset{\sim}{f}(\underset{\sim}{x}') = \underset{\sim}{\Omega}^{\infty}\cdot\underset{\approx}{C}(\underset{\sim}{x}') \quad ,$$

$$\underset{\sim}{v}(\underset{\sim}{x}) = \underset{=}{E}\cdot\underset{\sim}{x} - \frac{1}{8\pi\mu}\oint_S \underset{\sim}{f}(\underset{\sim}{x}')\cdot\underset{=}{I}(\underset{\sim}{x}-\underset{\sim}{x}')dS(\underset{\sim}{x}')$$

(3.10c)

$$\text{with } \underset{\sim}{f}(\underset{\sim}{x}') = \underset{=}{E}:\underset{\equiv}{M}(\underset{\sim}{x}') \quad ,$$

where the resistance tensors $\underset{=}{A}$, $\underset{=}{C}$ and $\underset{\equiv}{M}$ depend on the particle shape, lead to "approximate" Faxen laws for the force, torque and stresslet.

(3.11a) $$\underset{\sim}{F} = \oint_S \underset{\sim}{v}^{\infty}(\underset{\sim}{x}')\cdot\underset{=}{A}(\underset{\sim}{x}')dS - \underset{\sim}{U}\cdot\oint_S \underset{=}{A}(\underset{\sim}{x}')dS - \oint_S (\underset{\sim}{\omega}\times\underset{\sim}{x}')\cdot\underset{=}{A}(\underset{\sim}{x}')dS$$

(3.11b) $$\underset{\sim}{T} = \oint_S \underset{\sim}{v}^{\infty}(\underset{\sim}{x}')\cdot\underset{=}{C}(\underset{\sim}{x}')dS - \underset{\sim}{U}\cdot\oint_S \underset{=}{C}(\underset{\sim}{x}')dS - \oint_S (\underset{\sim}{\omega}\times\underset{\sim}{x}')\cdot\underset{=}{C}(\underset{\sim}{x}')dS$$

(3.11c) $$\underset{=}{S} = \oint_S \underset{\sim}{v}^{\infty}(\underset{\sim}{x}')\cdot\underset{\equiv}{M}(\underset{\sim}{x}')dS - \underset{\sim}{U}\cdot\oint_S \underset{\equiv}{M}(\underset{\sim}{x}')dS - \oint_S (\underset{\sim}{\omega}\times\underset{\sim}{x}')\cdot\underset{\equiv}{M}(\underset{\sim}{x}')dS \quad .$$

4. MODULF INTEGRATION: HI BETWEEN TWO STATIONARY SPHERES

The method of reflections and the Faxen laws of the previous sections are combined in this section to calculate the drag on two spheres of radii a and b, with centers at $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, held fixed in a uniform stream $\underset{\sim}{U}^{\infty}$. The drag on sphere 1 as calculated by standard reflection techniques in Happel and Brenner (1965) is:

(4.1)
$$\frac{\underset{\sim}{F}_1}{6\pi\mu a} = \Big\{\Big[1 - \frac{3}{4}\frac{b}{R} + \frac{9}{16}\frac{ab}{R^2} - \Big(\frac{1}{4}\lambda + \frac{27}{64}\lambda^2 + \frac{1}{4}\lambda^3\Big)\frac{a^3}{R^3}$$
$$+ \Big(\frac{3}{8}\lambda + \frac{81}{256}\lambda^2 + \frac{9}{8}\lambda^3\Big)\frac{a^4}{R^4}\Big](\underset{=}{\delta} - \underset{\sim\sim}{dd})$$
$$+ \Big[1 - \frac{3}{2}\frac{b}{R} + \frac{9}{4}\frac{ab}{R^2} + \Big(\frac{1}{2}\lambda - \frac{27}{8}\lambda^2 + \frac{1}{2}\lambda^3\Big)\frac{a^3}{R^3}$$
$$+ \Big(-\frac{3}{2}\lambda + \frac{81}{16}\lambda^2 + \frac{9}{4}\lambda^3\Big)\frac{a^4}{R^4}\Big]\underset{\sim\sim}{dd}\Big\}\cdot\underset{\sim}{U}^{\infty}$$

where $\underset{\sim}{d} = (\underset{\sim}{x}_2-\underset{\sim}{x}_1)/|\underset{\sim}{x}_2-\underset{\sim}{x}_1|$, $\lambda = b/a$ and $R = |\underset{\sim}{x}_2-\underset{\sim}{x}_1|$. The method presented here not only recovers this result, but solves the parallel and perpendicular cases simultaneously.

The zero-th reflection at each sphere yields the Stokes drag as the leading contribution on each sphere:

(4.2a) $$\underset{\sim}{F}_1^{(0)} = 6\pi\mu a\underset{\sim}{U}^{\infty}$$

$$\underset{\sim}{F}_2^{(0)} = 6\pi\mu b\underset{\sim}{U}^{\infty} \quad . \tag{4.2b}$$

The reflected fields are the Stokes disturbance fields:

$$\underset{\sim}{v}_1 = -6\pi\mu a\underset{\sim}{U}^{\infty}\cdot(1 + \frac{a^2}{6}\nabla^2)\,\frac{\underset{=}{I}(\underset{\sim}{x}-\underset{\sim}{x}_1)}{8\pi\mu} \tag{4.3a}$$

$$\underset{\sim}{v}_2 = -6\pi\mu b\underset{\sim}{U}^{\infty}\cdot(1 + \frac{b^2}{6}\nabla^2)\,\frac{\underset{=}{I}(\underset{\sim}{x}-\underset{\sim}{x}_2)}{8\pi\mu} \quad . \tag{4.3b}$$

The contributions from the first reflection at sphere 1, $\underset{\sim}{F}^{(1)}$, is obtained by applying the Faxen law (for the *force*) with $\underset{\sim}{v}_2$ as the incident field,

$$\underset{\sim}{F}_1^{(1)} = 6\pi\mu a(1 + \frac{a^2}{6}\nabla^2)\{-6\pi\mu b\underset{\sim}{U}^{\infty}\cdot(1 + \frac{b^2}{6}\nabla^2)\,\frac{\underset{=}{I}(\underset{\sim}{x}-\underset{\sim}{x}_2)}{8\pi\mu}\}\Big|_{\underset{\sim}{x}=\underset{\sim}{x}_1}$$

so that

$$\frac{\underset{\sim}{F}_1^{(1)}}{6\pi\mu a} = -\frac{3}{4}\,b\underset{\sim}{U}^{\infty}\cdot\{1 + \frac{(a^2+b^2)}{6}\nabla^2\}\,\underset{=}{I}(\underset{\sim}{x}-\underset{\sim}{x}_2)\}\Big|_{\underset{\sim}{x}=\underset{\sim}{x}_1} \tag{4.4}$$

($\nabla^4\underset{=}{I} = \underset{\sim}{0}$ since $\underset{=}{I}$ is a solution of the Stokes equations). The substitutions for $\underset{=}{I}$ and $\nabla^2\underset{=}{I}$ reduce equation (4.4) to

$$\begin{aligned}\frac{\underset{\sim}{F}_1^{(1)}}{6\pi\mu a} = \{&[-\frac{3}{4}\frac{b}{R} - (\frac{1}{4}\lambda + \frac{1}{4}\lambda^3)\,\frac{a^3}{R^3}](\underset{=}{\delta}-\underset{\sim}{d}\underset{\sim}{d}) \\ &+ [-\frac{3}{2}\frac{b}{R} + (\frac{1}{2}\lambda + \frac{1}{2}\lambda^3)]\underset{\sim}{d}\underset{\sim}{d}\}\cdot\underset{\sim}{U}^{\infty} \quad .\end{aligned} \tag{4.5}$$

The contribution from the second reflection at sphere 1, $\underset{\sim}{F}^{(2)}$, is obtained by applying the Faxen law with $\underset{\sim}{v}_{12}$ (which was created by reflecting $\underset{\sim}{v}_1$ at sphere 2) as the incident field. Thus the multipole expansion for $\underset{\sim}{v}_{12}$ is required accurate to $O(R^{-4})$:

$$v_{12i} = -F_{2j}^{(1)}\,\frac{I_{ij}(\underset{\sim}{x}-\underset{\sim}{x}_2)}{8\pi\mu} + (S_{2jk}^{(1)} + T_{2jk}^{(1)})\,\frac{I_{ij,k}(\underset{\sim}{x}-\underset{\sim}{x}_2)}{8\pi\mu} - Q_{2jk\ell}^{(1)}\,\frac{I_{ij,k\ell}(\underset{\sim}{x}-\underset{\sim}{x}_2)}{16\pi\mu}$$

and the contribution from the second reflection follows as

$$\frac{F^{(2)}_{2i}}{6\pi\mu a} = -F^{(1)}_{2j}\left(1+\frac{a^2}{6}\nabla^2\right)\frac{I_{ij}(\underset{\sim}{x}-\underset{\sim}{x}_2)}{8\pi\mu}\bigg|_{\underset{\sim}{x}=\underset{\sim}{x}_1}$$

$$+\left(S^{(1)}_{2jk}+T^{(1)}_{2jk}\right)\frac{I_{ij,k}(\underset{\sim}{x}-\underset{\sim}{x}_2)}{8\pi\mu}\bigg|_{\underset{\sim}{x}=\underset{\sim}{x}_1} \tag{4.6}$$

$$-\,Q^{(1)}_{2jk\ell}\frac{I_{ij,k\ell}(\underset{\sim}{x}-\underset{\sim}{x}_2)}{16\pi\mu}\bigg|_{\underset{\sim}{x}=\underset{\sim}{x}_1}+O(R^{-6}) \quad .$$

The moments in (4.6) must be simplified to obtain the final expression. $\underset{\sim}{F}^{(1)}_2$ is obtained by switching a and b in (4.5). After the substitution for $\underset{=}{I}$ and $\underset{\sim}{F}^{(1)}_2$, the first term in (4.6) reduces to

$$\left\{\left[\frac{9}{16}\frac{ab}{R^2}+\left(\frac{3}{8}\lambda+\frac{3}{16}\lambda^3\right)\frac{a^4}{R^4}\right](\underset{=}{\delta}-\underset{\sim}{d}\underset{\sim}{d})+\left[\frac{9}{4}\frac{ab}{R^2}-\left(\frac{3}{2}\lambda+\frac{3}{4}\lambda^3\right)\frac{a^4}{R^4}\right]\underset{\sim}{d}\underset{\sim}{d}\right\}\cdot\underset{\sim}{U}^{\infty} \quad . \tag{4.7a}$$

In the second term of (4.6), the dipole moments, $\underset{=}{S}$ and $\underset{=}{T}$ are needed accurate only to $O(R^{-2})$ since they are multiplied by the Stokes dipole which decays as R^{-2}. The Faxen laws for these moments yield:

$$S^{(1)}_{2jk}\frac{I_{ij,k}(\underset{\sim}{x}-\underset{\sim}{x}_2)}{8\pi\mu}\bigg|_{\underset{\sim}{x}=\underset{\sim}{x}_1} = -\frac{20}{3}\pi\mu b^3 e_{1jk}\frac{I_{ij,k}(\underset{\sim}{x}_2-\underset{\sim}{x}_1)}{8\pi\mu}$$

$$T^{(1)}_{2jk}\frac{I_{ij,k}(\underset{\sim}{x}-\underset{\sim}{x}_2)}{8\pi\mu}\bigg|_{\underset{\sim}{x}=\underset{\sim}{x}_1} = -4\pi\mu b^3\Omega_{1jk}\frac{I_{ij,k}(\underset{\sim}{x}_2-\underset{\sim}{x}_1)}{8\pi\mu}$$

where $e_{1jk}=\frac{1}{2}(v_{ij,k}+v_{1k,j})$ and $\Omega_{ijk}=\frac{1}{2}(v_{1k,j}-v_{1j,k})$ are the rate of strain and vorticity fields of the Stokes solution. The substitutions for $\underset{=}{e}_1$, $\underset{=}{\Omega}_1$ and the Stokes dipole, simplify these contributions as

$$\frac{15}{4}\lambda^3\frac{a^4}{R^4}\underset{\sim}{d}\underset{\sim}{d}\cdot\underset{\sim}{U}^{\infty} \tag{4.7b}$$

and

$$\frac{3}{4}\lambda^3\frac{a^4}{R^4}(\underset{=}{\delta}-\underset{\sim}{d}\underset{\sim}{d})\cdot\underset{\sim}{U}^{\infty} \tag{4.7c}$$

respectively.

The last term in (4.6) makes a contribution through an $O(R^{-1})$ term in the quadrupole moment $\underset{=}{Q}$:

$$Q^{(1)}_{2jk\ell}=\frac{b^2}{3}F^{(1)}_{2j}\delta_{k\ell}+O(R^{-3}) \quad .$$

The Faxen law for $\underset{\approx}{Q}$ is shown in Kim (1983) along with a recursive formula for the Faxen law for the n-th multipole moment. However, the leading order term in $\underset{\approx}{Q}$ can be deduced from its presence in the Stokes solution. After the substitutions for $\underset{\sim}{F}_2^{(1)}$ and $\nabla^2 \underset{\approx}{I}$ the last contribution simplifies to

$$\left\{\frac{3}{16} \lambda^3 \frac{a^4}{R^4} (\underset{=}{\delta}-\underset{\sim\sim}{dd}) - \frac{3}{4} \lambda^3 \frac{a^4}{R^4} \underset{\sim\sim}{dd}\right\}\cdot\underset{\sim}{U}^\infty \quad . \tag{4.7d}$$

The calculation of the contributions from the third and fourth reflection is easier since only the leading order terms are needed at each reflection (point force approximation). As the reflections are traced back to the ambient field $\underset{\sim}{U}^\infty$, each reflection contributes a factor of $O(R^{-1})$ from the decay in $\underset{=}{I}(\underset{\sim}{x}_1-\underset{\sim}{x}_2)$.

$$\begin{aligned}\frac{\underset{\sim}{F}_1^{(3)}}{6\pi\mu a} &= \left(-\frac{3}{4}b\right)^2\left(-\frac{3}{4}a\right)\underset{=}{I}^3(\underset{\sim}{x}_2-\underset{\sim}{x}_1)\cdot\underset{\sim}{U}^\infty \\ &= \left\{-\frac{27}{64}\lambda^2\frac{a^3}{R^3}(\underset{=}{\delta}-\underset{\sim\sim}{dd}) - \frac{27}{8}\lambda^2\frac{a^3}{R^3}\underset{\sim\sim}{dd}\right\}\cdot\underset{\sim}{U}^\infty\end{aligned} \tag{4.8}$$

$$\begin{aligned}\frac{\underset{\sim}{F}_1^{(4)}}{6\pi\mu a} &= \left(-\frac{3}{4}b\right)^2\left(-\frac{3}{4}a\right)^2 \underset{=}{I}^4(\underset{\sim}{x}_2-\underset{\sim}{x}_1)\cdot\underset{\sim}{U}^\infty \\ &= \left\{\frac{81}{256}\lambda^2\frac{a^4}{R^4}(\underset{=}{\delta}-\underset{\sim\sim}{dd}) + \frac{81}{16}\lambda^2\frac{a^4}{R^4}\underset{\sim\sim}{dd}\right\}\cdot\underset{\sim}{U}^\infty \quad .\end{aligned} \tag{4.9}$$

The contributions from the zero-th through the fourth reflections given in (4.2), (4.5), (4.7a-d), (4.10) and (4.11) sum to (4.1). As mentioned earlier, the method generates simultaneously the solutions for streams parallel ($\underset{\sim\sim}{dd}\cdot\underset{\sim}{U}^\infty$ terms) and perpendicular ($(\underset{=}{\delta} - \underset{\sim\sim}{dd})\cdot\underset{\sim}{U}^\infty$ terms) to the sphere-sphere axis. The method introduces even greater savings in the mobility problems of the *following section*. The parallel and perpendicular cases will be handled simultaneously, but more importantly, the method solves directly a specific mobility problem. For example, the motion of torque-free particles (e.g., sedimentation) will be solved by imposing the torque-free condition at each reflection.

The result will be the one of interest, and thus other subsidiary (and unwanted) problems will be avoided.

5. MODULE INTEGRATION: SEDIMENTATION OF TWO SPHERES

The problem of determing the motion of two torque-free spheres under the influence of external forces (e.g., sedimenation) can be considered the inverse of the problem discussed in the previous section (Brenner 1964a). Our immediate concern here is to show that the present method gives results that agree with the expressions for the mobility functions as deduced from the expressions given by Happel and Brenner (1965).

Consider two spheres and the geometry of the previous section. External forces $\underset{\sim}{F}_1$ and $\underset{\sim}{F}_2$ but no torques are imposed on each sphere. The objective is to express the translational and rotational velocities $\underset{\sim}{U}_1$, $\underset{\sim}{U}_2$ and $\underset{\sim}{\omega}_1$, $\underset{\sim}{\omega}_2$ in terms of $\underset{\sim}{F}_1$, $\underset{\sim}{F}_2$ and $\underset{\sim}{d}$. Since this form is not the one given in Happel and Brenner, theirs will be rearranged. We start with the general relation between the forces, torques, translational velocity and rotational velocity given by Brenner (1964a):

$$(5.1) \qquad \begin{bmatrix} \underset{\sim}{F}_1 \\ \underset{\sim}{F}_2 \\ \underset{\sim}{T}_1 \\ \underset{\sim}{T}_2 \end{bmatrix} = \begin{bmatrix} \underset{\approx}{A}_{11} & \underset{\approx}{A}_{12} & \underset{\approx}{\tilde{C}}_{11} & \underset{\approx}{\tilde{C}}_{12} \\ \underset{\approx}{A}_{21} & \underset{\approx}{A}_{22} & \underset{\approx}{\tilde{C}}_{21} & \underset{\approx}{\tilde{C}}_{22} \\ \underset{\approx}{C}_{11} & \underset{\approx}{C}_{12} & \underset{\approx}{B}_{11} & \underset{\approx}{B}_{12} \\ \underset{\approx}{C}_{21} & \underset{\approx}{C}_{22} & \underset{\approx}{B}_{21} & \underset{\approx}{B}_{22} \end{bmatrix} \begin{bmatrix} \underset{\sim}{U}_1 \\ \underset{\sim}{U}_2 \\ \underset{\sim}{\omega}_1 \\ \underset{\sim}{\omega}_2 \end{bmatrix}$$

The matrix elements are second order tensors and the usual rules for matrix operations are in effect. For the problem of interest, $\underset{\sim}{T}_1 = \underset{\sim}{T}_2 = \underset{\sim}{0}$, so that $\underset{\sim}{\omega}_1$ and $\underset{\sim}{\omega}_2$ may be eliminated from (5.1) to give:

$$(5.2) \qquad \begin{bmatrix} \underset{\sim}{F}_1(6\pi\mu a)^{-1} \\ \underset{\sim}{F}_2(6\pi\mu b)^{-1} \end{bmatrix} = \begin{bmatrix} \underset{\approx}{A}'_{11} & \underset{\approx}{A}'_{12} \\ \underset{\approx}{A}'_{21} & \underset{\approx}{A}'_{22} \end{bmatrix} \begin{bmatrix} \underset{\sim}{U}_1 \\ \underset{\sim}{U}_2 \end{bmatrix} , \quad \underset{\sim}{T}_1 = \underset{\sim}{T}_2 = 0 \ .$$

The symmetry in the two-sphere geometry leads to a further simplification -- each $\underset{\approx}{A}'_{\alpha\beta}(\alpha,\beta = 1 \text{ or } 2)$ can be written in terms of scalar functions for the parallel and

perpendicular cases, i.e.

$$\underset{=}{A}'_{\alpha\beta} = X_{\alpha\beta}(a/R,\lambda)\underset{\sim}{d}\underset{\sim}{d} + Y_{\alpha\beta}(a/R,\lambda)(\underset{=}{\delta}-\underset{\sim}{d}\underset{\sim}{d}) \quad . \tag{5.3}$$

Since the tensors commute, the inverse of the matrix in (5.2) can be written as

$$\begin{bmatrix} \underset{=}{A}'_{22}\cdot\underset{=}{D}^{-1} & -\underset{=}{A}'_{12}\cdot\underset{=}{D}^{-1} \\ -\underset{=}{A}'_{21}\cdot\underset{=}{D}^{-1} & \underset{=}{A}'_{11}\cdot\underset{=}{D}^{-1} \end{bmatrix} \tag{5.4}$$

with $\underset{=}{D} = \underset{=}{A}'_{11}\cdot\underset{=}{A}'_{22} - \underset{=}{A}'_{21}\cdot\underset{=}{A}'_{12}$. We now substitute Happel and Brenner's (1965) expressions (6-3.51) and (6-3.96) for X and Y:

$$X_{11} = 1 + \frac{9}{4}\frac{ab}{R^2} + \left(-\frac{3}{2}\lambda^2 + \frac{9}{4}\lambda^3\right)\frac{a^4}{R^4} + \ldots \tag{5.5a}$$

$$X_{12} = -\frac{3}{2}\frac{b}{R} + \left(\frac{1}{2}\lambda - \frac{27}{8}\lambda^2 + \frac{1}{2}\lambda^3\right)\frac{a^3}{R^3} - \frac{9}{4}\left(\lambda^2 + \frac{27}{8}\lambda^3 + \lambda^4\right)\frac{a^5}{R^5} + \ldots \tag{5.5b}$$

$$Y_{11} = 1 + \frac{9}{16}\frac{ab}{R^2} + \frac{3}{8}\left(\lambda + \frac{27}{32}\lambda^2\right)\frac{a^4}{R^4} + \ldots \tag{5.5c}$$

$$Y_{12} = -\frac{3}{4}\frac{b}{R} - \frac{1}{4}\left(\lambda + \frac{27}{16}\lambda^2 + \lambda^3\right)\frac{a^3}{R^3} - \frac{27}{64}\left(\lambda^2 + \frac{9}{16}\lambda^3 + \lambda^4\right)\frac{a^5}{R^5} + \ldots \quad . \tag{5.5d}$$

The expressions for X_{22}, X_{21}, Y_{22} and Y_{21} are obtained by switching a and b in (5.5a-d). The matrix elements in (5.4) can now be written as

$$\underset{=}{A}'_{22}\cdot\underset{=}{D}^{-1} = X_{22}(X_{11}X_{22} - X_{21}X_{12})^{-1}\underset{\sim}{d}\underset{\sim}{d} + Y_{22}(Y_{11}Y_{22} - Y_{21}Y_{12})^{-1}(\underset{=}{\delta}-\underset{\sim}{d}\underset{\sim}{d}) \tag{5.6}$$

and

$$-\underset{=}{A}_{12}\cdot\underset{=}{D}^{-1} = -X_{12}(X_{11}X_{22} - X_{21}X_{12})^{-1}\underset{\sim}{d}\underset{\sim}{d} - Y_{12}(Y_{11}Y_{22} - Y_{21}Y_{12})^{-1}(\underset{=}{\delta}-\underset{\sim}{d}\underset{\sim}{d}) \tag{5.7}$$

so that after some tedious algebra, the translational velocity can be written as:

$$\begin{aligned} \underset{\sim}{U}_1 = &\left\{\left[1 + \frac{0}{R^2} - \frac{15}{4}\lambda^3\frac{a^4}{R^4} + \ldots\right]\underset{\sim}{d}\underset{\sim}{d} + \left[1 + \frac{0}{R^2} + \frac{0}{R^4} + \ldots\right](\underset{=}{\delta}-\underset{\sim}{d}\underset{\sim}{d})\right\}\cdot\frac{\underset{\sim}{F}_1}{6\pi\mu a} \\ &+ \left\{\left[\frac{3}{2}\frac{b}{R} - \frac{1}{2}(\lambda+\lambda^3)\frac{a^3}{R^3} + \frac{0}{R^5} + \ldots\right]\underset{\sim}{d}\underset{\sim}{d}\right. \\ &\left. + \left[\frac{3}{4}\frac{b}{R} + \frac{1}{4}(\lambda+\lambda^3)\frac{a^3}{R^3} + \frac{0}{R^5} + \ldots\right](\underset{=}{\delta}-\underset{\sim}{d}\underset{\sim}{d})\right\}\cdot\frac{\underset{\sim}{F}_2}{6\pi\mu b} \quad . \end{aligned} \tag{5.8}$$

The zeros that appear in (5.8) are not accidental, but are consequences of the reflection process and the properties of the Oseen tensor, as shown in the following direct computation for $\underset{\sim}{U}_1$.

We now obtain (5.8) with the present method by writing $\underset{\sim}{U}_1$ as the sum of the contributions from all reflections as obtained from the Faxen law:

$$\underset{\sim}{U}_1^{(0)} = \underset{\sim}{F}_1(6\pi\mu a)^{-1}$$

$$\underset{\sim}{U}_1^{(1)} = \left(1 + \frac{a^2}{6}\nabla^2\right)\underset{\sim}{v}_2\big|_{\underset{\sim}{x}=\underset{\sim}{x}_1}$$

$$\underset{\sim}{U}_1^{(2)} = \left(1 + \frac{a^2}{6}\nabla^2\right)\underset{\sim}{v}_{12}\big|_{\underset{\sim}{x}=\underset{\sim}{x}_1} \quad .$$

Since $\underset{\sim}{v}_2$ is the Stokes solution, $\underset{\sim}{U}_1^{(1)}$ simplifies as:

$$\underset{\sim}{U}_1^{(1)} = \left(1 + \frac{a^2}{6}\nabla^2\right)\underset{\sim}{F}_2\cdot\left(1 + \frac{b^2}{6}\nabla^2\right)\frac{\underset{=}{I}(\underset{\sim}{x}-\underset{\sim}{x}_2)}{8\pi\mu}\Big|_{\underset{\sim}{x}=\underset{\sim}{x}_1}$$

(5.9)
$$= \frac{\underset{\sim}{F}_2}{6\pi\mu b}\cdot\left(\frac{3}{4}b\right)\left\{1 + \frac{(a^2+b^2)}{6}\nabla^2\right\}\underset{=}{I}(\underset{\sim}{x}-\underset{\sim}{x}_2)\big|_{\underset{\sim}{x}=\underset{\sim}{x}_1}$$

$$= \frac{\underset{\sim}{F}_2}{6\pi\mu b}\cdot\left\{\left[\frac{3}{2}\frac{b}{R} - \frac{1}{2}(\lambda+\lambda^3)\frac{a^3}{R^3}\right]\underset{\sim}{d}\underset{\sim}{d} + \left[\frac{3}{4}\frac{b}{R} + \frac{1}{4}(\lambda+\lambda^3)\frac{a^3}{R^3}\right](\underset{=}{\delta}-\underset{\sim}{d}\underset{\sim}{d})\right\} \quad .$$

The calculation for $\underset{\sim}{U}_1^{(2)}$ introduces a novel twist which turns out to be a labor saving feature in many mobility problems. Since the first reflection already contributed $\underset{\sim}{F}_1$, subsequent reflections cannot contain a monopole term. Furthermore, since the spheres are torque-free, the multipole expansion for any of the higher order reflection fields must lead off with the stresslet term:

(5.10)
$$\underset{\sim}{v}^R = S_{jk}\left(1 + \frac{b^2}{10}\nabla^2\right)\frac{I_{ij,k}(\underset{\sim}{x}-\underset{\sim}{x}_2)}{8\pi\mu} + \ldots$$

with

$$S_{jk} = \frac{20}{3}\pi\mu b^3\left(1 + \frac{b^2}{10}\nabla^2\right)e^I_{ij}\big|_{\underset{\sim}{x}=\underset{\sim}{x}_2} \quad .$$

When such reflection fields are used as incident fields at the next reflection, the leading

order term in the translational velocity at that next reflection will be of $O(R^{-3})$ smaller than the leading term in the translational velocity contributed by the previous reflection, i.e., $\underset{\sim}{U}_1^{(n+1)} = O(R^{-3})\underset{\sim}{U}_1^{(n)}$. The decay in the Stokes dipole contributes a factor of $O(R^{-2})$ and the stresslet is $O(R^{-1})$ smaller than the translational velocity of the preceding reflection. Therefore, the leading order terms in $\underset{\sim}{U}_1^{(2)}$ and $\underset{\sim}{U}_1^{(3)}$ are of $O(R^{-4})$ and $O(R^{-7})$ respectively.

$$\underset{\sim}{U}_1^{(2)} = \frac{20}{3}\pi\mu b^3(\underset{=}{e}_1\cdot\nabla)\cdot\frac{\underset{=}{I}(\underset{\sim}{x}-\underset{\sim}{x}_2)}{8\pi\mu}\bigg|_{\underset{\sim}{x}=\underset{\sim}{x}_1} = \frac{\underset{\sim}{F}_1}{6\pi\mu a}\cdot\left\{-\frac{15}{4}\lambda^3\frac{a^4}{R^4}\underset{\sim}{d}\underset{\sim}{d}\right\} \tag{5.11}$$

$$\underset{\sim}{U}_1^{(3)} = \frac{20}{3}\pi\mu b^3(\underset{=}{e}_{21}\cdot\nabla)n\,\frac{\underset{=}{I}(\underset{\sim}{x}-\underset{\sim}{x}_2)}{8\pi\mu}\bigg|_{\underset{\sim}{x}=\underset{\sim}{x}_1} = O(R^{-5})\cdot O(R^{-2}) \quad . \tag{5.12}$$

The translational velocities from these contributions sum to (5.8) and as mentioned earlier, the zero coefficients in the mobility functions are due to the weaker interactions beyond the first reflection.

The rotational velocity $\underset{\sim}{\omega}_1$ can be related to $\underset{\sim}{d}\times\underset{\sim}{F}_1$ by inserting (5.8) into expressions (6-3.97) of Happel and Brenner (1965),

$$\begin{aligned}\underset{\sim}{\omega}_1 = &-\underset{\sim}{d}\times\underset{\sim}{U}_1\left[\frac{9}{16}\frac{ab}{R^3} + \frac{3}{16}\left(\lambda + \frac{27}{16}\lambda^2 + \lambda^3\right)\frac{a^4}{R^5}\right] \\ &+ \underset{\sim}{d}\times\underset{\sim}{U}_2\left[\frac{3}{4}\frac{b}{R^2} + \frac{27}{64}\lambda^2\frac{a^3}{R^4} + \frac{9}{32}\left(\lambda^2 + \frac{27}{32}\lambda^3 + \lambda^4\right)\frac{a^5}{R^6}\right] \quad .\end{aligned} \tag{5.13}$$

The result is:

$$\begin{aligned}\underset{\sim}{\omega}_1 = &-\underset{\sim}{d}\times\underset{\sim}{F}_1(6\pi\mu a)^{-1}\left[\frac{0}{R^3} + \frac{0}{R^5} + \ldots\right] \\ &+ \underset{\sim}{d}\times\underset{\sim}{F}_2(6\pi\mu b)^{-1}\left[\frac{3}{4}\frac{b}{R^2} + \frac{0}{R^4} + \frac{0}{R^6} + \ldots\right] \quad .\end{aligned} \tag{5.14}$$

Again, the zero coefficients are due to the weaker interactions.

Our method of reflections sums the contributions from each reflection. Again, it would appear that factors of $O(R^{-3})$ will be introduced. However, the stresslet field is irrotational so that the interactions are even weaker (at most $O(R^{-5})$). Therefore,

$$\omega_1^{(1)} = \frac{1}{2}\,(\nabla\times v_2)$$

(5.15a)

$$= d\times F_2(6\pi\mu b)^{-1}\,\frac{3}{4}\,\frac{b}{R^2}$$

$$\omega_1^{(2)} = \frac{1}{2}\,(\nabla\times v_{12})\Big|_{x=x_1} = \frac{1}{2}\,\nabla\times\Big\{S_2^{(1)}:\frac{\nabla I(x-x_2)}{8\pi\mu} + \ldots\Big\}\Big|_{x=x_1}$$

(5.15b)

$$= d\times F_1(6\pi\mu a)^{-1}\Big[\frac{0}{R^5} + \ldots\Big]$$

$$\omega_1^{(3)} = \frac{1}{2}\,(\nabla\times v_{212})\Big|_{x=x_1} = \frac{1}{2}\,\nabla\times\Big\{S_2^{(2)}:\frac{\nabla I(x-x_2)}{8\pi\mu} + \ldots\Big\}\Big|_{x=x_1}$$

(5.5c)

$$= d\times F_2(6\pi\mu b)^{-1}\Big[\frac{0}{R^8} + \ldots\Big]$$

and the rotational velocity follows as

$$\omega_1 = -d\times F_1(6\pi\mu a)^{-1}\Big[\frac{0}{R^5} + \ldots\Big]$$

(5.16)

$$+ d\times F_2(6\pi\mu b)^{-1}\Big[\frac{3}{4}\,\frac{b}{R^2} + \frac{0}{R^4} + \frac{0}{R^6} + \frac{0}{R^8} + \ldots\Big] \quad .$$

In summary, the method reproduces the resistance and mobility functions for spheres given by Happel and Brenner (1965). Furthermore, the results are consistent with the recent and more extensive calculations of Jeffrey and Onishi (1983). In the following parts, the method will be applied to HI problems involving spheroid-spheroid interactions, spheroid-wall interactions and multi-particle (three-sphere) interactions. These calculations will lay the foundation for future investigations into the effect of multi-particle HI on the rheology of a suspension of nonspherical particles.

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NOTATION

a radius of sphere at x_1.

$\underset{\sim}{A}$ element of the resistance tensor.

b radius of sphere at x_2.

$\underset{\sim}{B}$ element of the resistance tensor.

c distance from center to foci.

$\underset{\sim}{C}$ element of the resistance tensor.

$\underset{\sim}{d}$ unit vector denoting orientation of spheroid axis.

e eccentricity of the spheroid.

$\underset{\sim}{E}$ rate of strain tensor.

$\underset{\sim}{F}$ force exerted on the particle by the fluid.

$\underset{\sim}{J}$ Oseen tensor defined by equation (2.2).

$\underset{\sim}{n}$ outward normal vector for a surface.

p pressure.

$\underset{=}{Q}$ quadrupole moment of the surface-force distribution.

R center to center separation between two spheres.

$\underset{\sim}{S}$ stresslet or symmetric part of the stress-dipole.

$\underset{\sim}{T}$ torque exerted on the particle by the fluid.

$\underset{\sim}{U}$ particle translational velocity.

$\underset{\sim}{v}$ velocity.

$\underset{\sim}{x}$ position vector.

$\underset{\sim}{x}'$ point on the surface $S(\underset{\sim}{x}')$.

X scalar function in the resistance tensor, parallel problem.

Y scalar function in the resistance tensor, perpendicular problem.

α constants in the Chwang-Wu singularity solutions.

γ constants in the Chwang-Wu singularity solutions.

$\underset{=}{\delta}$ identity tensor.

$\underset{=}{\varepsilon}$ alternating tensor.

λ ratio of sphere radii, b/a.

μ viscosity.

ξ vector denoting position on the spheroid axis.

$\underset{\sim}{\sigma}$ stress tensor.

$\underset{\sim}{\omega}$ particle angular velocity.

$\underset{\sim}{\Omega}$ vorticity.

$\underset{\approx}{\Omega}$ vorticity tensor.

Subscripts

1,2 refers to spheres at $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$.

i,j,k,ℓ,m indices used in the Einstein summation convention.

Superscripts

(n) denotes association with the n-th reflection.

∞ ambient field.

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A modular method for calculating hydrodynamic interactions between particles in low-Reynolds-number flow has been constructed by using multipole expansion solutions for the reflection field. The approach is made possible by the use of Faxen laws in relating the multipole moment to the incident field. The method is illustrated and checked by recalculating known expressions for the resistance and mobility tensors for two spheres. The method can be readily generalized to handle three-particle (or n-particle) interactions as shown in a following paper. New forms of the Faxen laws for

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ABSTRACT (continued)

prolate spheroids are given and will form the basis for other papers on spheroid-spheroid and spheroid-wall hydrodynamic interactions. The important result is that "first-reflection" solutions can be readily calculated even in cases where the ambient velocity field is obtained by a numerical procedure. These results, as asymptotic (far-field) solutions, furnish a check for more robust codes. In addition, they are important on their own since they provide crucial information for the renormalization theory used in suspension rheology.